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INVERTIBLE LIFTINGS OF MEASURABLE TRANSFORMATIONS

BY

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ABSTRACT

A relatively simple construction is given to show that, under mild hypotheses, a given measurable nonsingular surjection of a measure space S onto a measure space S' can be lifted to an invertible (injective) one, θ, between larger measure spaces; and in the endomorphic case (in which $S = S'$) the transformation θ can also be made an endomorphism.

1. Introduction

A well-known theorem of Rokhlin [7], recently extended by Silva [8], asserts that (under suitable conditions) a given endomorphism of a measure space can be lifted to a "natural extension" that is invertible (i.e., injective). This enables many classical results on invertible transformations to be applied to noninvertible ones. However, Rokhlin's construction, even as simplified by Silva, is complicated. The object of the present paper is to give a much simpler construction, involving nothing more than multiplication by the unit interval I. Our main results (Theorems 1 and 2, §3.2 below) assert, roughly, that (under mild hypotheses), given a measurable nonsingular surjection T of a measure space S onto a measure space S', there is an invertible measurable nonsingular surjection θ of the product measure space $(S\backslash N) \times I$ onto $(S'\backslash N') \times I$, where N and N' are null sets, such that θ induces T by projection. Further (Theorem 2) if $S = S'$ we may take $N' = N$ (so that θ too is an endomorphism). This addition to Theorem 1 does not appear to be trivial; at any rate, the proof of Theorem 2 takes considerably more work than that of Theorem 1.

It should be emphasized that the extension θ of T produced in the present paper is quite different from the "natural" one of Rokhlin and Silva, and may lack

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some of the latter's properties; in particular, it is not clear to what extent ergodicity and entropy can be preserved. In fact, the present construction as it stands does *not* preserve ergodicity in general (see the example in §6.6 below), and it probably does not preserve entropy either (though it is not hard to see that the extension θ will automatically have entropy at least that of T). The difficulty is that the construction has a good deal of arbitrariness in the choice of Borel isometries (ϕ, in particular) in §4.6, and a simple-minded choice is inadequate (6.6). It would be very desirable to know whether a sophisticated choice of these isometries could enable the present method to preserve both ergodicity and entropy.

Though I have not been able to settle this, the simplicity of the present method seems to justify it. Basically, the method exploits the fact that the measure spaces I and I^2 are isometric (isomorphic under a measure-preserving bijection). To apply this, we use the planar representation theorem of Rokhlin [6] and the author [3,4], in a form essentially due to Graf and Mauldin [1]. The map T induces a disintegration of S; an appropriate isomorphism converts S into a planar model M, and converts T to the projection π (from the plane to the x-axis). Identifying S with M, we obtain (after discarding suitable null sets) isomorphic disintegrations

$$\theta \circ \varpi : S \times I \to S' \qquad \text{and}$$

$$\varpi : S' \times I \to S',$$

where ϖ is projection onto the first factor. In the endomorphic case we represent T by π followed by an isomorphism. In both cases, as a matter of technical convenience, we use $I^- = [0,1)$ rather than I; the two measure spaces are, of course, isometric. By using the linear interval $\mathbf{R}^+ = [0,\infty)$ instead of I we could even make θ an isometry in Theorem 1 (Corollary 5.2).

All these considerations require that the measure spaces involved be standard (after possibly discarding null sets); we are essentially dealing with Polish spaces with (completed, σ-finite, not necessarily diffuse) measures. An extension of the results to nonseparable spaces (e.g., large products) would be interesting, but seems to present considerable difficulty. However, it can be shown that, from the measure-algebraic point of view, the analogs of Theorems 1 and 2 (for σ-finite measure algebras) are valid without any separability conditions, provided I is replaced by a suitable power $I^{\mathcal{M}}$ (with product measure).

We begin by discussing, in §2, several types of nonsingularity (of many-to-one measurable transformations) and the relations between them, since the later results depend on them. The main results (Theorems 1 and 2) are stated in §3, after the necessary definitions. Section 4 contains a detailed description of the planar model disintegration (in a slightly modified form, adjusted for simplicity and convenience) and some of its properties. Theorem 1 follows easily in §5; and §6 gives the proof of Theorem 2, after a further lemma (6.1).

2. Nonsingularity

The notion of nonsingularity (for many-to-one transformations) receives little explicit definition in the literature — perhaps because, in the "nice" cases usually considered, all reasonable definitions agree (cf. Proposition 2.3 below). We shall need to distinguish between some alternative definitions, which we accordingly give in full.

2.1. Notation. A *measure space* $\mathcal{S} = (S, \mathcal{M}, \mathcal{N}, \mu)$ consists of a set S, a σ-field $\mathcal{M}$ of subsets of S, and a non-negative countably additive measure μ on $\mathcal{M}$; $\mathcal{N}$ denotes the σ-ideal of μ-null sets. We often abbreviate $\mathcal{S}$ to (S, μ) or even S. Except where the contrary is explicitly stated, we assume that μ is σ-finite and complete (so that $\mathcal{M} \supset \mathcal{N}$). Each subset X of S defines a subspace $\mathfrak{X} = (X, \mathcal{M}_X, \mathcal{N}_X, \mu_X)$ of $\mathcal{S}$, where $\mathcal{M}_X = \{A \cap X : A \in \mathcal{M}\}$, $\mathcal{N}_X = \{N \cap X : N \in \mathcal{N}\}$ and $\mu_X(A \cap X) = \mu^*(A \cap X)$, μ^* denoting the outer measure defined by μ. We shall nearly always have $X \in \mathcal{M}$ here, so μ_X will be just the restriction of μ, and in fact we are usually concerned only with the case in which X is of full measure (i.e., $S \backslash X \in \mathcal{N}$).

A *transformation* (or map) T, from a measure space $\mathcal{S}$ to a measure space $\mathcal{S}' = (S', \mathcal{M}', \mathcal{N}', \mu')$, is a map $T: S \to S'$. It is (μ, μ')-*measurable*, or just "measurable" for short, if $T^{-1}(\mathcal{M}') \subset \mathcal{M}$, and *negatively nonsingular* (n.n.s. for short) or (μ, μ')-n.n.s. when emphasis is needed, provided $T^{-1}(\mathcal{N}') \subset \mathcal{N}$. (The term "nonsingular" might be preferable here, but has been pre-empted with a different meaning; cf. [8, p. 1].) We call T an *isomorphism* if it is a bijection of S onto S' such that both T and T^{-1} are measurable and n.n.s.; if, further, T is measure-preserving, T is an *isometry*.†

We say that the measure space $\mathcal{S}$ is *Polish* provided S is a complete separable metric space (or, what comes to the same thing to within Borel isometry, a separable metrizable absolutely Borel set) with completed Borel measure. A measure space isomorphic to a Polish one is "*standard*".

†For the purposes of this §2, isomorphic measure spaces are equivalent, and we are really concerned only with their equivalence classes. These could be called "measurable spaces", by analogy with "metrizable" and "metric" spaces, but again the term has been pre-empted.

2.2. Let $T: S \to S'$ be an arbitrary map from a measure space $\mathbb{S}$ to a measure space $\mathbb{S}'$. We say that T is *weakly nonsingular* (w.n.s. for short) provided that it is measurable and satisfies, for all $H \in \mathcal{M}'$,

$$T^{-1}(H) \in \mathcal{N} \Leftrightarrow T(T^{-1}(H)) \ (= H \cap T(S)) \in \mathcal{N}'. \tag{1}$$

(Note that this implies that T is n.n.s.) We say that T is "n.s." providing it is measurable and satisfies (1) for *all* $H \subset S'$. (For surjections, at least, this agrees with the usual use of the term "nonsingular"; cf. [8, p. 1].) And T is *strongly measurable* provided it is n.s. and satisfies, for all $H \subset S'$,

$$T^{-1}(H) \in \mathcal{M} \Leftrightarrow T(T^{-1}(H)) \in \mathcal{M}'. \tag{2}$$

One easily checks that here "n.s." could be replaced by "w.n.s.".

Though w.n.s., n.s. and strong measurability are different in general, even for bijective transformations, it turns out that, in the case of principal interest (when the measure spaces are standard and T is surjective), all three notions coincide (Proposition 2.4 below). The property most naturally involved in our considerations below is "n.s.", but it seems likely that strong measurability may be needed in more general situations.

Note that every measurable T can be converted into a w.n.s. map by discarding a suitable null set N from S and a suitable measurable set G (not necessarily null) from S'. (Merely take G to be a member of the largest measure class, in $\mathbb{S}'$, for which $T^{-1}(G) \in \mathcal{N}$, and put $N = T^{-1}(G)$.)

2.3. We shall later make use of the following version of a folk-theorem:

PROPOSITION. *Suppose T is a w.n.s. surjective transformation from $\mathbb{S}$ to $\mathbb{S}'$. Then there is a (σ-finite, complete) measure μ_1 on $\mathcal{M}$, equivalent to μ, such that T^{-1} is measure-preserving in the sense: for all $H \in \mathcal{M}'$,*

$$\mu_1(T^{-1}(H)) = \mu'(H).$$

SKETCH OF PROOF. Without loss of generality, μ is finite. Put $\mathcal{M}_1 = T^{-1}(\mathcal{M}')$; then $\mathcal{M}_1 \subset \mathcal{M}$ and $T^{-1}(\mathcal{N}') = \mathcal{M}_1 \cap \mathcal{N}$. Define a (not necessarily complete) measure ν on $\mathcal{M}_1$ by

$$\nu(T^{-1}(H)) = \mu'(H) \qquad (H \in \mathcal{M}');$$

then ν and μ are equivalent (have the same null sets) on $\mathcal{M}_1$. Extend ν to the desired measure μ_1 on $\mathcal{M}$ by defining, for each $A \in \mathcal{M}$, f_A to be the conditional expectation $E(A \mid \mathcal{M}_1)$, and defining

$$\mu_1(A) = \int_S f_A \, d\nu \qquad (A \in \mathcal{M}).$$

2.4. Proposition. *Let $T: S \to S'$ be a Borel measurable, n.n.s. transformation from a Polish measure space $\mathcal{S}$ to a Polish measure space $\mathcal{S}'$. Then $T(S) \in \mathcal{M}'$, and the following are equivalent*:

(1) *T is w.n.s.*

(2) *T is n.s.*

(3) *T is strongly measurable.*

Proof. $T(S)$ is analytic [2, p. 478], hence measurable. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. We deduce (1) $\Rightarrow$ (3) from the following lemma:

Lemma. *Let T be a w.n.s. transformation from an arbitrary measure space $\mathcal{S}$ to an arbitrary measure space $\mathcal{S}'$, and suppose each $A \in \mathcal{M}$ has a kernel K (that is, $K \subset A$, $K \in \mathcal{M}$ and $A \backslash K \in \mathcal{N}$) such that $T(K) \in \mathcal{M}'$. Then T is strongly measurable.*

Taking the lemma for granted temporarily, suppose (1) holds in the proposition. Given $A \in \mathcal{M}$, take a Borel subset B of S such that the symmetric difference $A \Delta B \in \mathcal{N}$, and take a Borel null set N containing $A \Delta B$. Define $K = A \backslash N$, and apply the lemma.

2.5. Proof of Lemma 2.4. We must prove, for all $H \subset S'$,

(i) if $T^{-1}(H) \in \mathcal{M}$ then $T(T^{-1}(H)) \in \mathcal{M}'$,

(ii) if $T^{-1}(H) \in \mathcal{N}$ then $T(T^{-1}(H)) \in \mathcal{N}'$,

the converse implications following trivially from the fact that T is measurable and n.n.s. The first step is to prove

(iii) $T(S) \in \mathcal{M}'$.

For take $K \in \mathcal{M}$ such that $S \backslash K \in \mathcal{N}$ and $T(K) \in \mathcal{M}'$, and put $L = S' \backslash T(K)$; then $T^{-1}(L) \subset S \backslash K$, so $T^{-1}(L) \in \mathcal{N}$. Weak nonsingularity gives $T(T^{-1}(L)) \in \mathcal{N}'$; that is, $L \cap T(S) \in \mathcal{N}'$. Thus $T(S) = T(K) \cup (L \cap T(S)) \in \mathcal{M}'$.

Calling a set of the form $T^{-1}(L)$, where $L \subset S'$, an "inverse set" for short, we note that if A (in the hypothesis of the lemma) is an inverse set, then K can be required to be an inverse set also; we merely replace it by $K_1 = T^{-1}(T(K))$ without spoiling its other properties.

Now suppose $A = T^{-1}(H) \in \mathcal{M}$. As just noted, there is an inverse set, say $K_1 = T^{-1}(K')$, such that $K_1 \subset A$, $A \backslash K_1 \in \mathcal{N}$ and $T(K_1) = K' \in \mathcal{M}'$. Similarly, since $S \backslash A$ is also an inverse set, there is an inverse set $E = T^{-1}(E')$ such that

$E \subset S\backslash A$, $(S\backslash A)\backslash E \in \mathcal{N}$, and $T(E) = E' \in \mathcal{M}'$. Write $L = S\backslash E$, $L' = T(L) = T(S)\backslash E'$; we have

$$K' \subset T(A) \subset L' \subset T(S) \quad \text{and} \quad T^{-1}(L'\backslash K') = L\backslash K_1 \in \mathcal{N}.$$

Here $L'\backslash K' \in \mathcal{M}'$, so the weak nonsingularity of T gives $L'\backslash K' \in \mathcal{N}'$. *A fortiori* $T(A)$ differs from L' by a null set and is therefore measurable, proving (i).

Finally, suppose $A = T^{-1}(H) \in \mathcal{N}$. By (i), $T(A) \in \mathcal{M}'$, so the w.s.n. property gives $T(A) \in \mathcal{N}'$, establishing (ii).

2.6. We conclude this section by observing that the "nearly standard" situation can, in a sense, be reduced to the "standard" with a strongly measurable transformation.

Proposition. *Suppose T is a measurable n.n.s. transformation from $\mathcal{S}$ to $\mathcal{S}'$, and that there are null sets $N \in \mathcal{N}$ and $N' \in \mathcal{N}'$ such that the subspaces (determined by) $S\backslash N$ and $S'\backslash N'$ (of $\mathcal{S}$ and $\mathcal{S}'$, respectively) are (isomorphic to) Polish measure spaces. Then there are subsets N^* of S and G of S' such that $N^* \supset N$, $G \supset N'$, $N^* \in \mathcal{N}$, $T^{-1}(G) \subset N^*$, and such that the subspaces $S\backslash N^*$ and $S'\backslash G$ are Polish and the restriction of T to $S\backslash N^*$ is a Borel measurable strongly measurable surjection onto $S'\backslash G$.*

Remark. In general, N^* will not be an inverse set, and G will not be null, though we can arrange for G to be null if (for instance) T is w.n.s. to start with.

Sketch of Proof. By discarding null sets we may assume $\mathcal{S}'$ is Polish, and (as remarked in 2.2) we can further discard a null set from S and a subset G of S' so that (the restriction of) T is w.n.s. After further discarding of null Borel sets we obtain a situation to which Proposition 2.4 can be applied, making the transformation strongly measurable.

3. Liftings and quotients

3.1. Let $T: \mathcal{S} = (S, \mathcal{M}, \mathcal{N}, \mu) \twoheadrightarrow (S', \mathcal{M}', \mathcal{N}', \mu') = \mathcal{S}'$ be a surjective strongly measurable map of measure spaces. A *lifting* of T is an isomorphism θ between measure spaces $\mathcal{R} = (R, \mathcal{P}, \mathcal{Q}, \nu)$ and $\mathcal{R}' = (R', \mathcal{P}', \mathcal{Q}', \nu')$ for which there are strongly measurable surjections $\sigma: R \twoheadrightarrow S$ and $\sigma': R' \twoheadrightarrow S'$ such that the diagram

$$\begin{tikzcd} R \arrow[r, two heads, shift left, "\theta"] \arrow[d, two heads, "\sigma"'] & R' \arrow[l, dashed, shift left] \arrow[d, two heads, "\sigma'"] \\ S \arrow[r, two heads, "T"'] & S' \end{tikzcd}$$

commutes. In this situation we also say that T is a "quotient" of θ. The inverse maps produce a dual commuting diagram of (algebraically) isomorphic null-set-preserving maps of the σ-fields of measurable sets (not in general surjective),

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow{\theta^{-1}} & \mathcal{M}' \\ {\scriptstyle \sigma^{-1}}\uparrow & & \uparrow{\scriptstyle (\sigma')^{-1}} \\ \mathcal{P} & \xleftarrow[T^{-1}]{} & \mathcal{P}' \end{array}.$$

Clearly the notions of "lifting" and "quotient" are invariant under isomorphisms of the measure spaces involved. In particular, they are unaffected by replacement of the given measures by equivalent ones.

In the "endomorphic" case, in which $\mathcal{S} = \mathcal{S}'$ and $\mathcal{R} = \mathcal{R}'$, the notion of "quotient", as defined here, coincides (at least when the measure spaces are Polish) with "factor" in the usual terminology (cf. [8]).

One more piece of notation is needed. The unit interval $[0,1]$, with Lebesgue measure λ, is denoted by I, and its subspace $[0,1)$ by I^-.

3.2. Now we can state the main theorems of the paper.

THEOREM 1. *Let $\mathcal{S} = (S, \mathcal{M}, \mathcal{N}, \mu)$ and $\mathcal{S}' = (S', \mathcal{M}', \mathcal{N}', \mu')$ be Polish measure spaces; let $T: S \twoheadrightarrow S'$ be a strongly measurable and Borel measurable surjection; and let $N \in \mathcal{N}$ and $N' \in \mathcal{N}'$ be given. Then there exist Borel sets $V \subset S \setminus N$ and $V' \subset S' \setminus N'$, of full μ – and μ' – measure, respectively, such that $V' = T(V)$, and a Borel isomorphism θ between the product measure spaces $V \times I$ and $V' \times I$, that is also an isomorphism with respect to the product measures $\mu \times \lambda$, $\mu' \times \lambda$, such that θ is a lifting of the restriction $T \mid V$, with commuting diagram*

$$\begin{array}{ccc} V \times I & \overset{\theta}{\rightleftarrows} & V' \times I \\ {\scriptstyle \varpi}\downarrow & & \downarrow{\scriptstyle \varpi'} \\ V & \xrightarrow[T \mid V]{} & V' \end{array}$$

where ϖ and ϖ' are the projections to the first coordinates. If, further, T^{-1} is measure-preserving (that is, if $\mu(T^{-1}(H)) = \mu'(H)$ for all $H \in \mathcal{M}'$), then θ can be required to be an isometry that is "measure-preserving on fibres" in the sense that, for each $x' \in V'$, the map $\theta: \varpi^{-1}(T^{-1}(x') \times I) \to \{x'\} \times I$ is an isometry (preserving λ).

THEOREM 2. *In Theorem 1, if T is an endomorphism (that is, if $\mathcal{S} = \mathcal{S}'$) we may require $V' = V$, so that θ becomes an automorphism (and $\varpi = \varpi'$).*

In both theorems, the ability to discard the given null sets N and N' allows the theorems to cover somewhat more general situations; for instance, they apply to analytic (rather than merely Polish) spaces. And it allows us (by Proposition 2.6) to assume throughout that T is Borel measurable.

The proofs of both theorems involve passing to an isomorphic planar model, which we next describe.

4. The adjusted planar model

4.1. *The set.* It is convenient to make slight modifications in the planar representation of a measurable subfield as given in [4] and [1] (see Figure 1). The basic situation is unchanged; we have a Borel subset M of $\mathbf{R}^2$, with a completed Borel measure μ, and the projection $M' = \pi(M)$ on the x-axis $\mathbf{R}$, with a completed Borel measure μ', satisfying certain conditions. To state them, we use the following no-

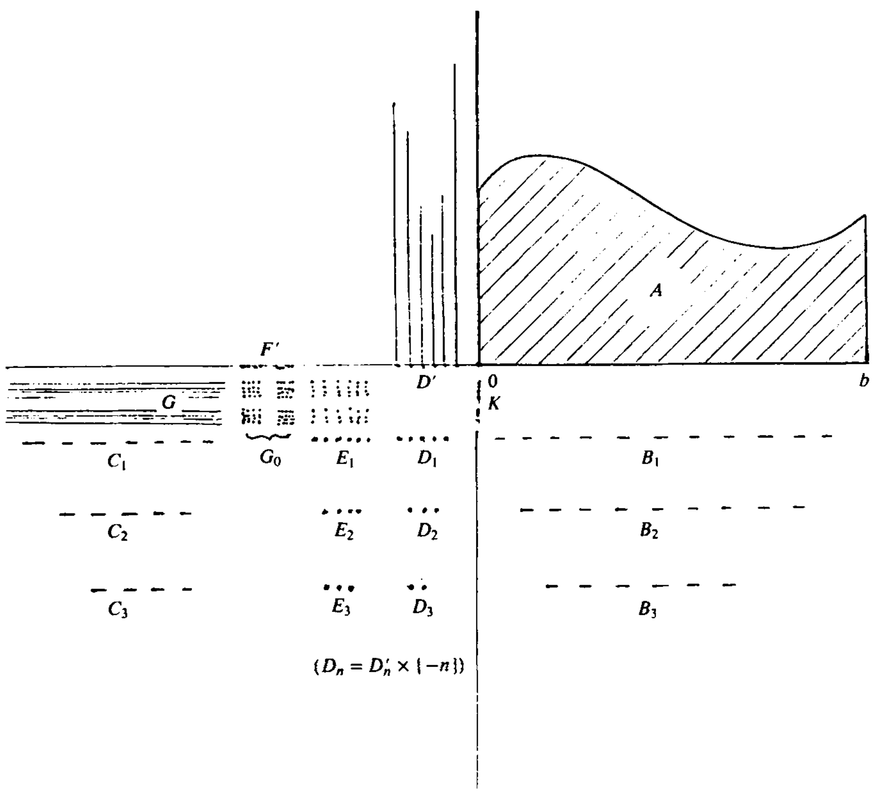

Fig. 1.

tation. If $Y \subset \mathbf{R}^2$, its projection $\pi(Y)$ is denoted by Y', and the restriction $\pi \mid Y$ is written π_Y (so $\pi_Y^{-1}(H) = \pi^{-1}(H) \cap Y$ for $H \subset Y'$). The "fiber" $(\pi_Y)^{-1}\{x\}$, where $x \in Y'$, is written Y_x.

We require that M' is a Borel subset of $\mathbf{R}$ and that π_M is measurable and n.n.s. with respect to μ and μ'. (In the case of interest to us, π_M will be strongly measurable, but this is not required initially.) There are completed Borel measures ν_x on M_x $(x \in M')$, forming a (strict) disintegration for the map $\pi_M : M \twoheadrightarrow M'$. Further, $M = P \cup G$ where $P \cap G = \varnothing$, with P (the "principal set") expressed as a union of pairwise disjoint sets (possibly empty),

$$P = A \cup B \cup C \cup D \cup E,$$

where:

(1) A is the ordinate set $\{(x,y) : 0 \leq x < b,\ 0 \leq y < f(x)\}$ of a strictly positive Borel measurable extended-real function f on $[0,b)$, for some $b \geq 0$ (possibly infinite).

(2) $B = \bigcup\{B_n : n \in \mathbf{N}\}$ where $B_n = B_n' \times \{-n\}$, B_n' being a Borel subset of $[0,b)$ of positive linear Lebesgue measure λ (if not empty), and where $B_1' \supset B_2' \supset \cdots$. $(\mathbf{N} = \{1,2,3,\ldots\}.)$

(3) Similarly, $C = \bigcup\{C_n : n \in \mathbf{N}\}$ where $C_n = C_n' \times \{-n\}$, C_n' being a Borel subset of $(-\infty,-3)$ of positive Lebesgue measure (if not empty) and where $C_1' \supset C_2' \supset \cdots$, subject to the further requirement that $(-\infty,-3)\setminus C_1'$ has infinite Lebesgue measure.

(4) $D = \{(d,y) : 0 \leq y < g(d),\ d \in D'\} \cup \{(d_n,-n) : n \in \mathbf{N},\ d_n \in D_n'\}$, where D' is a countable (possibly finite or empty) subset of $(-1, 0)$, each $g(d)$ is positive (possibly infinite), and $D' \supset D_1' \supset D_2' \supset \cdots$.

(5) $E = \bigcup\{E_n : n \in \mathbf{N}\}$ where $E_n = \{(e,-n) : e \in E_n'\}$, where E_n' is a countable subset of $(-2,-1)$ and $E_1' \supset E_2' \supset \cdots$.

(6) G, the "garbage set", is a Borel subset of the product $((-\infty,-3) \cup E_1' \cup F') \times K$, where F' is a λ-null Borel subset of $(-3,-2)$ and K is a λ-null Cantor subset of $(-1,0)$.

4.2. *The measures.* The measure μ on M is defined as follows. On A, μ is plane Lebesgue measure λ^2; on each B_n and C_n, μ is equivalent to linear Lebesgue measure λ, with Radon–Nikodym derivative $d\mu/d\lambda$ everywhere finite, strictly positive and Borel measurable. On D, μ coincides with λ on each interval $\{(d,y) : 0 \leq y < g(d)\}$, while the points $(d_n,-n)$ are atoms of μ, as are also the points of E. Finally, μ on G is 0 (in agreement with λ^2).

The measure μ' on M' is λ, except on $D_1' \cup E_1'$, each point of which is a

μ'-atom (of finite positive measure). (This is simpler than in the previous planar models, in which the nonatomic part of μ' was merely *equivalent* to λ.)

The "slice-measures" $\nu_x, x \in M'$, are defined as follows. On $M_x \cap (\{x\} \times [0,\infty))$, where $x > -1$, ν_x is λ. Each point $(x,-n)$ of $B_n \cup C_n$ is a ν_x-atom of weight $(d\mu/d\lambda)(x,-n)$ (cf. the definition of μ). The points $(d_n,-n)$ of D are also ν_x-atoms where $x = d_n$, of weight $\nu_x(x,-n) = \mu(x,-n)/\mu'(x)$. On the interval $\{(d,y) : 0 \leq y < g(d)\}$, $d \in D'$, of D, the ν_d-measure is λ divided by the constant $\mu'(d)$. Similarly, each point of E is a ν_x-atom with

$$\nu_x(x,-n) = \mu(x,-n)/\mu'(x) \qquad (x \in E'_n).$$

Finally, for $x \in G' = \pi(G)$, $\nu_x(G_x) = 0$.

The (strict) disintegration requirement, that for each measurable $A \subset M$ we have $\mu(A) = \int_{M'} \nu_x(A_x)\, d\mu'(x)$, is easily seen to hold.

4.3. *The adjustments.* The planar model just described is almost identical with those in [1] and [4], but with slightly different placement. On comparing it with [4], the "atoms" of measure 0 have been eliminated; the sequences $\{B'_n : n \in \mathbf{N}\}$, $\{C'_n : n \in \mathbf{N}\}$ and $\{D'_n : n \in \mathbf{N}\}$ are now decreasing; the "almost ordinate set" of [4] has become an ordinate set (as in [1]), and the parts of the "garbage set" below the ordinate set or below D' have been absorbed into the intervals above them. And G' has been moved from $(-3,-2)$ to a subset of $(-\infty,-3) \cup E' \cup F'$, allowing the measure on $M' \setminus (D'_1 \cup E'_1)$ to be simplified to λ.

All these modifications are easily produced, e.g. from [1], either by routine horizontal and vertical isometries ("positioning") as in [4], or by removing some null vertical cylinders (transferring them to the garbage set). In particular, vertical isometries are used to absorb null sets, via the parametrization theorem of Mauldin [5].

4.4. Proposition. *Let $T : \mathcal{S} \twoheadrightarrow \mathcal{S}'$ be a strongly measurable and Borel measurable surjection of Polish measure spaces. Then the disintegration determined by T is isometric, under a Borel isomorphism, to the disintegration provided by an adjusted planar model in which the property* (6) *of* 4.1 *is strengthened to*

(6′) *G is a Borel subset of* $(C'_1 \cup E'_1 \cup F') \times K$

(*where F' and K are as in* 4.1(6)).

Proof. The disintegration induced by T is isometric to a planar model disintegration, as in [4] or [1], which (by the process described in 4.3) is in turn isometric to the adjusted planar model of 4.1 and 4.2. We now have that π_M is n.s. (because T is), which implies that $\mu'(G' \cap ((-\infty,-3) \setminus C'_1)) = 0$. Thus $G \setminus ((C'_1 \cup$

$E'_1 \cup F') \times K)$ is a null set both horizontally and vertically, and (as in 4.3) it can be eliminated by absorption into $F' \times K$.

REMARK. Conversely, it is easy to see that (6′) implies that π_M is strongly measurable. We refer to this as the "strongly measurable case".

4.5. *The essential garbage set.* Suppose we are in the strongly measurable case. Define

$$G_0 = M \setminus \pi_M^{-1}(P') = \pi_M^{-1}(M' \setminus P').$$

Since $P' = [0, b) \cup C'_1 \cup D'_1 \cup E'_1$, a Borel set, we see that G_0 is a Borel "cylinder" subset of M. Also $G_0 \subset G$ and

$$G'_0 = \{x \in M' : \nu_x(M_x) = 0\}.$$

From condition (6′) above, $G'_0 \subset F'$ and consequently $\mu'(G'_0) = 0$. We note also the obvious properties

$$G'_0 = M' \setminus P'; \quad P' = \{x \in M' : \nu_x(M_x) > 0\}; \quad \mu'(M' \setminus P') = 0 = \mu(M \setminus P).$$

We define $R = \pi_M^{-1}(P') = M \setminus G_0$; this is a Borel set of full μ-measure in M, and $R \supset P$ and $R_x = M_x$ for all $x \in P'$. We also write $\Theta = \{(x, y) : x \in P', 0 \le y < \nu_x(M_x)\}$; thus $\Theta_x = [0, \nu_x(M_x))$ for all $x \in P'$.

4.6. With the notation just introduced (and with I^- denoting $[0, 1)$ with Lebesgue measure) we have:

PROPOSITION. *Assume π_M is strongly measurable. There is a Borel isomorphism θ of $R \times I^-$ onto Θ that is a $(\mu \times \lambda, \mu' \times \lambda)$ isometry, such that*

(1) *θ is a lifting (in the sense of 3.1) of the projection π_R of R onto P', the associated maps σ and σ' being the projections ϖ and ϖ' to the first coordinates (so that $\varpi' = \pi_\Theta = \pi \mid \Theta$).*

(2) *For each $x \in P'$ $(= R')$, the restriction $\theta \mid R_x \times I^-$ is a $(\nu_x \times \lambda, \lambda)$ isometry of $R_x \times I^-$ onto Θ_x.*

(Thus θ is a Borel isometry, in the sense of [4], between the disintegrations induced, respectively, by

$$\varpi \circ \pi_R : R \times I^- \twoheadrightarrow (P', \mu')$$

and

$$\varpi' : \Theta \twoheadrightarrow P'.)$$

PROOF. First we verify that π_R is strongly measurable. By Proposition 2.4 it suffices to show that π_R is n.s.; and (since $\mu(M \setminus R) = 0$) this follows easily from the fact that π_M is n.s. It is also easy to see that the projection maps ϖ and ϖ' are strongly measurable.

The next step is to define θ on $(A \cup B) \times I^-$. Fixing $x \in [0,b)$ for the moment, we note that

$$M_x = (\{x\} \times [0,f(x)) \cup \{(x,-n) : n \in \mathbf{N}(x)\}$$

for a suitable subset $\mathbf{N}(x)$ of $\mathbf{N}$ (possibly finite or empty). We arrange for θ to map $M_x \times I^-$ isometrically onto

$$\mathcal{O}_x = \{x\} \times [0, \nu_x(M_x))$$

as follows. Suppose first $f(x) = \infty$. Writing $J_n = [n-1, n)$ $(n \in \mathbf{N})$, so that $J_1 = I^-$, we have

$$M_x \times I^- = \bigcup \{\{x\} \times (J_n \times I^-) : n \in \mathbf{N}\} \cup \bigcup \{(x,-n) \times I^- : n \in \mathbf{N}(x)\}.$$

Fix a (λ^2, λ) Borel isometry ϕ, independent of x, of the square $J_1 \times I^-$ onto the interval J_1. We define θ to send $\{x\} \times (J_1 \times I^-)$ onto J_1 by means of ϕ, then to send the interval $(x,-1) \times I^-$ onto $[1, 1 + \nu_x(x,-1))$ by a linear map, then $\{x\} \times (J_2 \times I^-)$ onto $[1 + \nu_x(x,-1), 2 + \nu_x(x,-1))$ by a translate of ϕ, then $(x,-2) \times I^-$ onto

$$[2 + \nu_x(x,-1), \quad 2 + \nu_x(x,-1) + \nu_x(x,-2))$$

by a linear map, and so on. When $n \in \mathbf{N} \setminus \mathbf{N}(x)$ we put $\nu_x(x,-n) = 0$, so that the corresponding image is just the empty set. Thus θ maps $M_x \times I^-$ isometrically onto the linear interval $[0, f(x) + \Sigma_n \nu_x(x,-n)) = [0,\infty) = [0, \nu_x(M_x))$.

If instead $f(x) < \infty$, the construction is similar but simpler; we send the rectangle $\{x\} \times [0,f(x)) \times I^-$ to the interval $[0,f(x))$ by a Borel (λ^2, λ) isometry, by applying the scaling factor $f(x)$ to the map ϕ, and then adjoin the images of the sets $(x,-n) \times I^-$ as successive intervals of the correct lengths.

Now let x vary in $[0,b)$. The resulting map θ takes $(A \cup B) \times I^-$ onto $\mathcal{O} \cap \pi^{-1}[0,b)$ isometrically, and it is a Borel isomorphism too, because f and the functions ν_x are Borel measurable.

Similarly, we define θ on $D \times I^-$. The procedure is essentially the same as before, the function f being replaced by the function g. The image set is $\mathcal{O} \cap \pi^{-1}(D')$.

We must still define θ on $(C \cup E \cup (G \backslash G_0)) \times I^-$. Observing that $G \backslash G_0 = U \cup V$ where

$$U = G \cap ((-\infty, -3) \times \mathbf{R}) = G \cap \pi^{-1}(C_1') \quad \text{and}$$

$$V = G \cap ((-2,-1) \times \mathbf{R}) = G \cap \pi^{-1}(E_1'),$$

we begin by defining θ on $(C \cup U) \times I^-$. Here $U \subset C_1' \times K$, so each U_x is ν_x-null; we combine U with C_1 by the following device. Fix a Borel isomorphism ξ of $K \times I^-$ onto K, and define a Borel isomorphism η of $C_1' \times K \times I^-$ onto $C_1' \times K$ by: $\eta(x,y,t) = (x, \xi(y,t))$, so that we have $\eta(G_x \times I^-) \subset \{x\} \times K$ for each $x \in C_1'$. Noting that $C_1 = C_1' \times \{-1\}$, we extend η to a Borel isomorphism ψ of $C_1' \times (\{-1\} \times K) \times I^-$ onto $C_1' \times (I^- \cup K)$ by putting (for $x \in C_1'$, $y \in K$ and $t \in I^-$)

$$\psi(x,-1,t) = (x,t),$$

$$\psi(x,y,t) = \eta(x,y,t).$$

Thus $\psi(U \times I^-) = W$, say, is a Borel subset of $C_1' \times K$, and ψ maps $(C_1 \times I^-) \cup (U \times I^-)$ onto $(C_1' \times I^-) \cup W$ in a first-coordinate-preserving way that also preserves the measure $(\nu_x \times \lambda)$ on each x-fiber $(x \in C_1')$ — the contributions of $U \times I^-$ and of W $(\subset C_1' \times K)$ to this measure both being 0. Using Mauldin's parametrization theorem [5], as in [1], we obtain a further first-coordinate-preserving Borel isomorphism ρ of $(C_1' \times I^-) \cup W$ onto $C_1' \times I^-$ that also preserves the fiber-measures.

To define θ on $(C_1 \cup U) \times I^-$ we first apply the Borel isomorphism $\rho \circ \psi$, which replaces $(C_1 \cup U) \times I^-$ by $C_1' \times I^-$, and then map this to Θ by sending the point (x,t) (where $x \in C_1'$ and $t \in I^-$) to the point $(x, t\nu_x(x,-1))$.

To define θ on $(C_2 \cup C_3 \cup \cdots) \times I^-$, we proceed as for $A \cup B$; that is, for each $x \in C_n'$, where $n = 2,3,\ldots$, the interval $\{(x,-n)\} \times I^-$ is mapped linearly onto the linear interval

$$\{x\} \times \left[\sum_{i=1}^{n-1} \nu_x(x,-i),\ \sum_{j=1}^{n} \nu_x(x,-j)\right).$$

Thus θ maps each "fiber" $(C \cup U)_x \times I^-$ onto the interval $\{x\} \times [0, \nu_x(M_x)) = \Theta_x$ in a measure-preserving way.

To define θ on $(E \cup V) \times I^-$ the procedure is similar but simpler (as E_1' is countable); $V \times I^-$ is combined with $E_1 \times I^-$, and $E_2, E_3, \ldots$ are treated just like $C_2, C_3, \ldots$.

The combined map $\theta : R \times I^- \twoheadrightarrow \Theta$ is a Borel isomorphism, preserving the measures on the spaces and on the fibers, and it constitutes a lifting of π_R as required. (That θ is strongly measurable is a trivial consequence of its being an isometry.)

4.7. COROLLARY. *Proposition* 4.6 *continues to hold if* $R \times I^-$ *is replaced by* $R \times \mathbf{R}^+$ *and* $\mathcal{O}$ *by* $P' \times \mathbf{R}^+$.

(Here $\mathbf{R}^+$ denotes the interval $[0,\infty)$.)

For let $\mathcal{O}_1, \mathcal{O}_2, \ldots$ be pairwise disjoint isometric copies of $\mathcal{O}$, and (as in 4.6) let J_n denote the interval $[n-1,n)$. Proposition 4.6 gives isometries θ_n ($n \in \mathbf{N}$) of $R \times J_n$ onto $\mathcal{O}_n$; these combine to a single Borel isometry of $R \times \bigcup_n J_n$ onto $\bigcup_n \mathcal{O}_n$, and this in turn is Borel isometric to $P' \times \mathbf{R}^+$, as can be seen as follows. Partition $\mathcal{O}$ into Borel sets (say) $Q_1, Q_2, \ldots$, with every $\nu_x(Q_m) < \infty$. This induces a corresponding partition of $\mathcal{O}_n$ into sets, say Q_{nm}. Reassemble the Q_{nm}'s into a single sequence, and for each x map the x-fibers of $Q_{n_1m_1}, Q_{n_2m_2}, \ldots$ successively onto consecutive intervals over x of the appropriate lengths.

REMARK. Since $\mathbf{R}$ and $\mathbf{R}^+$ are isometric, we could of course replace $\mathbf{R}^+$ by $\mathbf{R}$ in the Corollary.

5. Proof of Theorem 1

5.1. Assume the hypotheses of Theorem 1 (3.2). From Proposition 2.3 there is a (σ-finite, complete Borel) measure μ_1 on $\mathcal{M}$, equivalent to μ, such that T^{-1} is (μ_1, μ') measure-preserving. Replacing μ by μ_1 does not alter the content of the theorem, and enables us to assume that T^{-1} is measure-preserving—that is, for all $L \in \mathcal{M}'$, $\mu(T^{-1}(L)) = \mu'(L)$.

Next we dispose of the exceptional null sets N, N'. We may assume (by enlarging them) that they are null Borel sets. Put $H = T^{-1}(S' \setminus N') \setminus N$; this is a Borel set of full μ-measure in S. And $T(H)$ is an analytic [2, p. 478], hence measurable, subset of $S' \setminus N'$, of full μ'-measure in S'. Thus there is a Borel set $N'' \in \mathcal{N}'$ containing $S' \setminus T(H)$. Define $W = H \cap T^{-1}(S' \setminus N'')$, a Borel set of full μ-measure in S, and put $T(W) = W'$, so that $W' = S' \setminus N''$. The subspaces W of $\mathcal{S}$, W' of $\mathcal{S}'$, are (to within Borel isometry) Polish measure spaces, and $T \mid W$ is a strongly measurable, Borel measurable surjection of W onto W'. To simplify matters we write S for W, S' for W' and T for $T \mid W$ in what follows; in effect, the null sets N and N' have been discarded.

By Proposition 4.4, the disintegration of S ($= W$) over S' ($= W'$) induced by T is Borel isometric to an adjusted planar model as in §4, with S corresponding to M, S' to M' and T to π_M, and with π_M strongly measurable. Using the notation of 4.5, define $V_1 = R = M \setminus G_0$ and $V_1' = P' = M' \setminus G_0'$, and apply Proposition

4.6. This produces an isometric fiber-respecting Borel isomorphism θ of $V_1 \times I^-$ onto $\mathcal{O}$. Because of the measure-preserving property of T^{-1} we here have $\nu_x(M_x) = 1$ for almost all $x \in V_1'$, say for all $x \in V_1' \backslash N_1$ where N_1 is a μ'-null Borel subset of V_1'. Define $V' = V_1' \backslash N_1$, $V = \pi_M^{-1}(V')$, and consider the restriction of θ to $V \times I^-$. This in effect replaces $\mathcal{O}$ by $V' \times I^-$ (since $\nu_x(M_x) = 1$ for all $x \in V'$), and $\theta \mid V \times I^-$ provides the lifting required by the theorem, in view of the fact that I^- and I are Borel isometric.

5.2. Corollary. *In Theorem* 1, *θ can be extended to an isometry of $V \times \mathbf{R}^+$ onto $V' \times \mathbf{R}^+$, maintaining its other properties.*

For Corollary 4.7 enables us to modify the preceding argument accordingly.

6. Proof of Theorem 2

The key step is the following Lemma:

6.1. Lemma. *Let $\mathcal{S} = (S, \mathcal{M}, \mathcal{N}, \mu)$ be a Polish measure space, let $T: S \twoheadrightarrow S'$ be a Borel measurable and strongly measurable surjection, and let $N_0 \in \mathcal{N}$ be given. Then there is a Borel subset V of S, of full μ-measure, such that* (i) $V \cap N_0 = \varnothing$, (ii) $T(V) = V$, (iii) *the quotient disintegration defined by $T \mid V: V \twoheadrightarrow V$ is isomorphic to an adjusted planar model, as in* §4, *in which $G = \varnothing$ and $\nu_x(V_x) = 1$ for all $x \in V$.*

Proof. Without loss of generality, μ is finite and N_0 is Borel. Consider the "quotient map" $p: S \twoheadrightarrow S/T = Q$, assigning to each $x \in S$ its equivalence class $p(x) = [x] = T^{-1}(T(x))$. There is a bijection q of Q onto S given by

$$q(p(x)) = T[x] = T(p(x));$$

thus $q \circ p = T$, and with the usual "abuse of notation" one writes $q = T \circ p^{-1}$. We use q to define a σ-field $\mathcal{R}$ of subsets of Q by: $\mathcal{R} = q^{-1}(\mathcal{M}) = \{q^{-1}(H): H \in \mathcal{M}\}$, and we give $\mathcal{R}$ the measure μ_1 induced by q from μ; that is,

$$\mu_1(K) = \mu(q(K)), \qquad K \in \mathcal{R}, \tag{1}$$

so that the map $q: (Q, \mathcal{R}, \mu_1) \twoheadrightarrow (S, \mathcal{M}, \mu)$ is an isometry.

We note the following properties:

$$\text{If } L \subset Q \text{ then } p^{-1}(L) = T^{-1}(q(L)); \tag{2}$$

$$\{T^{-1}(H): H \in \mathcal{M}\} = \{p^{-1}(K): K \in \mathcal{R}\}; \tag{3}$$

and, from the strong measurability of T,

(4) $$\mathcal{R} = \{K \subset Q : p^{-1}(K) \in \mathcal{M}\}.$$

From (4), the map $p: (S, \mathcal{M}) \twoheadrightarrow (Q, \mathcal{R})$ is measurable; we use it to define a second measure μ_2 on $\mathcal{R}$ by

(5) $$\mu_2(K) = \mu(p^{-1}(K)), \qquad K \in \mathcal{R}.$$

It is easy to see that the (finite) measures μ_1 and μ_2, though different in general, have the same null sets and are thus equivalent.

6.2. Now consider the (strict) disintegration induced by the map $p: (S, \mathcal{M}, \mu) \twoheadrightarrow (Q, \mathcal{R}, \mu_2)$. This is isometric to an adjusted planar model disintegration $\pi_M : M \twoheadrightarrow M'$, as in §4, in which (because p is measure-preserving) $\nu_x(M_x) = 1$ for almost all $x \in M'$. Making some further adjustments, as in 4.3, we arrange that $\nu_x(M_x) = 1$ for all $x \in M' \setminus G_0'$. (Of course, $\nu_x(M_x) = 0$ when $x \in G_0'$.) Without risk of confusion, we use the same symbol μ to denote the measure on M; the measure μ' on M' corresponds to μ_2.

The isometric Borel isomorphism between the disintegrations $p: (S, \mu) \twoheadrightarrow (Q, \mu_2)$ and $\pi_M : (M, \mu) \twoheadrightarrow (M', \mu')$ takes q to a (Borel) isomorphism of measure spaces (not, in general, an isometry), say $\rho : M' \to M$, and thus (because $T = q \circ p$) it takes T to $\rho \circ \pi_M$, which we denote by τ. Thus τ is a strongly measurable map from (M, μ) to (M, μ). From 6.1(3), T and p have the same measurable inverse sets; hence τ and π_M likewise have the same measurable inverse sets, which we shall refer to simply as "inverse sets".

We say that a subset N of M is "fully null" provided that, for all $x \in M'$, $\nu_x(N_x) = 0$. The "essential projection" of a Borel set $H \subset M$, namely $\{x \in M' : \nu_x(N_x) \neq 0\}$, is denoted by $\mathrm{ep}(H)$, and the "cylinder" on it, $\pi_M^{-1}(\mathrm{ep}(H))$, is denoted by $\mathrm{e}(H)$. With this notation, the following properties are clear.

(1) If H is a Borel inverse set, then $\tau(H)$ is Borel; if further H is μ-null, or of full μ-measure, then so is $\tau(H)$.

(2) If N is fully null, then $N \subset A \cup D^+ \cup G$, where $D^+ = \{(x,y) \in D : y \geq 0\}$.

(3) If N is fully null and H is an inverse set contained in $R (= M \setminus G_0)$, then $\pi_M(H) = \pi_M(H \setminus N)$, and hence $\tau(H) = \tau(H \setminus N)$.

(4) If H is a Borel subset of M, then
   (a) $\mathrm{ep}(H)$ is Borel, and is μ'-null if and only if $\mu(H) = 0$;
   (b) $H \setminus \mathrm{e}(H)$ is fully null; and
   (c) if $\mu(H) = 0$, then $\mathrm{e}(H)$ is a μ-null Borel inverse set.

6.3. We shall define, recursively, Borel inverse sets $V_1, V_2, \ldots$, of full μ-measure, and Borel fully null inverse sets $Z_1, Z_2, \ldots$, with $Z_n \subset V_n$, such that $V_n \supset \tau(V_n) \supset V_{n+1} \backslash Z_n$ $(n = 1, 2, \ldots)$. To start the induction, the given Borel null set N_0 of S corresponds to a Borel null set, which we also denote by N_0, contained in M. Put

$$N_1 = N_0 \cup e(N_0),$$
$$N_1^* = G_0 \cup e(N_0) \cup \bigcup \{\tau^{-n}(G \cup N_1) : n = 1, 2, \ldots\} \quad \text{and}$$
$$V_1 = M \backslash N_1^*.$$

Thus N_1^* is a null Borel inverse set (note that $G_0 = \pi_M^{-1}(G_0')$ is an inverse set), so that V_1 is a Borel inverse set of full μ-measure. The complementary inverse sets V_1 and N_1^* have complementary τ-images $\tau(V_1)$ and $\tau(N_1^*)$; both of these are analytic [2, p. 478], and hence both are Borel (2, p. 486]. Since τ is n.s., $\tau(N_1^*)$ is μ-null and $\tau(V_1)$ has full μ-measure. An easy calculation gives

$$\tau(N_1^*) \supset N_1^* \cup N_1 \cup G \supset N_0, \tag{1}$$

so that

$$\tau(V_1) \subset V_1 \cap (M \backslash G) = V_1 \cap P \quad \text{and} \tag{2}$$

$$\tau(V_1) \cap N_0 = \varnothing; \tag{3}$$

also

$$\tau^{-1}(N_1^*) \subset N_1^*. \tag{4}$$

Put $W_1 = V_1 \backslash \tau(V_1)$. One checks that

$$W_1 = \tau(N_1^*) \backslash N_1^* \subset \tau(N_1^*).$$

Put $W_1^* = e(W_1) \cup \bigcup \{\tau^{-n}(W_1 \cup e(W_1)) : n = 1, 2, \ldots\}$; this is a null Borel inverse set, and $\tau(W_1^*) \supset W_1^*$. Now define $V_2 = V_1 \backslash W_1^*$, a Borel inverse set of full measure, and $Z_1 = W_1 \backslash e(W_1)$, a fully null Borel inverse set. It is a routine matter to verify that $V_2 \supset \tau(V_2)$ and $\tau(V_1) \supset V_2 \backslash Z_1$.

When V_k and Z_{k-1} have been defined (for some $k \geq 2$), put $W_k = V_k \backslash \tau(V_k)$. Here V_k is a Borel inverse set of full measure; hence $\tau(V_k)$ is also Borel and of full measure, so W_k is null. Put

$$W_k^* = e(W_k) \cup \bigcup \{\tau^{-n}(W_k \cup e(W_k)) : n = 1, 2, \ldots\};$$

this is a null Borel inverse set, and $\tau(W_k^*) \supset W_k^*$. Define $V_{k+1} = V_k \setminus W_k^*$, $Z_k = W_k \setminus e(W_k)$; these are Borel inverse sets, with V_{k+1} of full measure and Z_k fully null, and we have $V_{k+1} \supset \tau(V_{k+1})$ and $\tau(V_k) \supset V_{k+1} \setminus Z_k$.

This completes the inductive process. Now define

$$V_\infty = \bigcap\{V_n : n = 1,2,\ldots\}, \quad V = \tau(V_\infty), \quad \text{and } Z = \bigcup\{Z_n : n = 1,2,\ldots\}.$$

Thus V_∞ is an inverse set, both V_∞ and V are Borel sets of full measure, and Z is a fully null Borel set. Since each V_n is an inverse set, we have $V = \bigcap\{\tau(V_n) : n = 1,2,\ldots\}$ and hence $V_\infty \supset V \supset V_\infty \setminus Z$, so that

$$\tau(V_\infty) = V \supset \tau(V) \supset \tau(V_\infty \setminus Z).$$

Now $V_\infty \subset V_1 = M \setminus N_1^* \subset M \setminus G_0$, and therefore (by 6.2(3)) $\tau(V_\infty) = \tau(V_\infty \setminus Z)$. Hence $V = \tau(V)$.

6.4. As observed above, $V_\infty \cap G_0 = \varnothing$; also V_∞ and G_0 are inverse sets, so $(V_\infty)' \cap G_0' = \varnothing$. Thus if $x \in (V_\infty)'$ we have $\nu_x((V_\infty)_x) = \nu_x(M_x) = 1$. But we have just seen that $V_\infty \supset V \supset V_\infty \setminus Z$, where Z is fully null. Thus $\nu_x(V_x) = 1$ for all $x \in (V_\infty)'$, and *a fortiori* for all $x \in V'$.

Now consider the restriction $\tau | V : V \twoheadrightarrow V$, regarded as a map from a subspace of M to a subspace of M'. It very nearly provides an "adjusted planar model" representation of the corresponding restriction of T, in the sense of §4, the sets $A \cap V$, $B \cap V$, and so on, taking on the roles of the sets $A, B, \ldots$, for M. In particular (from 6.2(2)) the construction makes $G \cap V = \varnothing$, so the resulting "garbage set" will be empty. Some further adjustments are needed, because in replacing M by V we have discarded a null Borel inverse set from M, and its null Borel image from M'. This results in the following departures from the requirements of §4:

(a) The set $A \cap V$ projects, not necessarily onto an interval $[0, b)$, but onto a Borel subset of it of full (Lebesgue) measure.

(b) In replacing A and D by $V \cap A$ and $V \cap D$, we may have removed some null subsets of vertical intervals.

To remedy (a), we apply an obvious isometry ("horizontal displacement"), adjusting ρ accordingly, to reconvert $\pi_M(A \cap V)$ to $[0, b)$. As for (b), insofar as D is concerned we can restore the mixing subsets by means of suitable "vertical displacements". For A, we use Mauldin's parametrization theorem [5] (cf. [1]) to fill out the missing null set in a first-coordinate-preserving Borel-and-measure-preserving way. Since these adjustments do not spoil the "$\nu_x(V_x) = 1$" property, the Lemma is proved (with the understanding that, to save notation, we have used the same symbol V to denote both the subset of M, constructed above, and the corresponding subset of S).

6.5. Now assume the hypotheses of Theorem 2; that is, $T: S \twoheadrightarrow S$ is a strongly measurable, Borel measurable, surjection of the Polish space $\mathcal{S} = (S, \mathcal{M}, \mathcal{N}, \mu)$ onto itself, and null sets $N \in \mathcal{N}$, $N' \in \mathcal{N}$, are given. Apply Lemma 6.1, taking $N_0 = N \cup N'$, and obtaining the set $V \subset S \setminus N_0$ of full measure described in the Lemma. We discard the invariant null set $S \setminus V$, and denote the (strongly measurable, Borel measurable) surjection $T \mid V$ by τ. As in 6.2, τ and the quotient map p that it induces produce the same disintegration, which (after an isometric isomorphism) we identify with an adjusted planar model disintegration $\pi_M: M \twoheadrightarrow M'$ in which we may assume $M = V$, $p = \pi_M$, $M' = V'$ ($= V$, but the notation is clearer if we distinguish between V as domain and V as range); also (as in 6.2) $\tau = \rho \circ \pi_M$ where ρ $\big($corresponding to $q \mid V\big)$ is an isomorphism (not, in general, measure-preserving) of V onto itself. From the Lemma we have $G = \varnothing$ and hence (in the notation of 4.5) $G_0 = \varnothing$ and therefore $R = M = P$ here. Also we have $\nu_x(M_x) = \nu_x(V_x) = 1$ for all $x \in M' = V'$, so that the set $\mathcal{O}$ of 4.6 is here just $V' \times I^-$. The measure μ' on M' is (by construction) μ_2 $(= \mu \circ p^{-1})$. Hence Proposition 4.6 gives a Borel-isomorphic $(\mu \times \lambda,\ \mu_2 \times \lambda)$ isometric lifting (say) θ^* of π_M, with first-coordinate projections as associated maps, as in the commutative diagram

$$\begin{tikzcd} V \times I^- \arrow[r, two heads, "\theta^*"] \arrow[d, "\varpi"'] & \mathcal{O} = V' \times I^{-1} \arrow[l, dashed, shift left] \arrow[d, "\varpi'"] \\ V \arrow[r, two heads, "\pi_M"'] & P' = V' \end{tikzcd}$$

in which, for each $x \in M'$ $(= V)$, $\theta^* \mid V_x \times I^-$ is a $(\nu_x \times \lambda, \lambda)$ isometry of $V_x \times I^-$ onto $\{x\} \times I^-$.

We combine this with the (trivial) commutative diagram

$$\begin{tikzcd} V' \times I^- \arrow[r, two heads, "\rho \times i"] \arrow[d, "\varpi"'] & V \times I^{-1} \arrow[d, "\varpi'"] \\ V' \arrow[r, two heads, "\rho"'] & V \end{tikzcd}$$

(where i denotes the identity map on I^-), noting that $\tau = \rho \circ \pi_M$, and see that the composition $(\rho \times i) \circ \theta^*$ provides the lifting θ required for Theorem 2.

6.6. *An example.* As remarked in the introduction, the construction used here does not automatically guarantee that, in Theorem 2, the lifted automorphism θ

will be ergodic if the given endomorphism T is ergodic. The following example illustrates this, even in the measure-preserving case.

Consider $H = I^{\aleph_0} = \prod_{n=1}^{\infty} I_{-n}$, with product Lebesgue measure (each I_{-n} being a copy of I), with T defined as the shift to the right on H; that is,

$$T(\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}) = (\ldots, \,.\,, x_{-n}, \ldots, x_{-2}).$$

Thus T is ergodic and T^{-1} is measure-preserving. Regard H as $H' \times I$, where I refers to the last coordinate. The quotient map p induced by T (as in 6.1) is then the projection onto H'. Because H' is isometric with I, we can regard H (to within isomorphism) as I^2, and we have a simple "adjusted plane model" with π_M = projection onto the first coordinate (i.e., H') and A $(= I^2)$ as the only ingredient of the model. Now $T = \rho \circ \pi_M$ (6.2), so ρ is the "shift" from H' to $H' \times I = H$, taking

$$(\ldots, x_{-n}, \ldots, x_{-2}) \quad \text{to} \quad (\ldots, \,.\,, x_{-n}, \ldots, x_{-3}, x_{-2}).$$

We have (from the end of 6.5) $\theta = (\rho \times i) \circ \theta^*$, where (from 4.6) θ^* sends $\{x\} \times [0,1] \times I$ to $\{x\} \times [0,1]$ by

$$\theta^*(x,y,z) = (x, \phi(y,z)) \qquad (x \in H';\, y,z \in I),$$

where ϕ is a fixed (but arbitrary) Borel isometry of (I^2, λ^2) onto (I, λ). (We may, and do, disregard the difference between the isometric spaces I^- and I here.) Accordingly, θ is the automorphism of H given by

$$\theta(\ldots, x_{-2}, x_{-1}, y, z) = (\ldots, \,.\,, x_{-2}, x_{-1}, \phi(y,z)).$$

Nothing prevents a "bad" choice of ϕ, say such that $\phi(I \times [0,1/2]) = [0,1/2]$. Then the set of all points of H with last coordinate $\leq 1/2$ is a nontrivial set invariant under θ^{-1}, so that θ will not be ergodic.

However, at least in the present case, there is also a "good" ϕ available, as follows. Write H^* for $I^{\aleph_0}$ regarded as $\prod_{n=1}^{\infty} I_n$; since H^* is Borel isometric to (I, λ) we can (to within isomorphism) define ϕ by

$$\phi(y;z) = \phi(y; z_1, z_2, \ldots) = (y, z_1, z_2, \ldots) \in H^* \approx I,$$

and then the resulting θ is defined on $\prod_{n=-\infty}^{\infty} I_n$ and sends

$$(\ldots, x_{-2}, x_{-1}, y, z_1, z_2, z_3, \ldots) \quad \text{(with } y \text{ in the zeroth place)}$$

to

$$(\ldots, x_{-2}, x_{-1}, y, z_1, z_2, \ldots,) \quad \text{(with } x_{-1} \text{ in the zeroth place)}.$$

That is, θ is the usual Bernoulli shift on $\prod_{-\infty}^{\infty} I_n$, and is ergodic. In this case, too, θ has the same entropy as T, namely ∞. (In general, all that seems to hold is $h(\theta) \geq h(T)$.)

As remarked in the introduction, it would be very interesting to know whether such "good" isometries (ϕ and the others in 4.6) are always available to preserve ergodicity and entropy.

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